SOME PLANE ADIABATIC IDEAL GAS FLOWS CONTAINING SHOCKS

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We obtain the solution describing adiabatic flows of an ideal gas characterized by the two parameters a and b such that

$$[a] = L^{m+1}T^{-1}, \quad [b] = ML^{-2-2m}$$

where m is arbitrary (m > 0). This solution permits the construction of flows containing shocks.

1. Assume that in the region y > 0 an ideal (i.e., without viscosity and thermal conductivity) perfect gas travels parallel to the Ox-axis and has the following parameters:

$$p \equiv 0, \ v \equiv 0, \ u = u_0(y) = ay^{-m}, \ \rho = \rho_0(y) = by^{2m-1} \tag{1.1}$$

and passes through a normal shock. The conditions on the shock have the form [1]

$$u = \frac{\gamma-1}{\gamma+1}u_0, \quad v = 0, \quad \rho = \frac{\gamma+1}{\gamma-1}\rho_0, \quad p = \frac{2}{\gamma+1}\rho_0 u_0^2 \tag{1.2}$$

In view of the presence of the pressure gradient in the direction of the Oy-axis, the flow behind the normal shock is described by the system of equations [2]

$$\frac{\partial}{\partial\eta}\frac{v}{u} + v\frac{\partial}{\partial\xi}\frac{v}{u} + p\frac{\partial}{\partial\xi}\frac{1}{\rho u} = 0, \quad \frac{\partial}{\partial\eta}\frac{1}{p} + \frac{\partial}{\partial\xi}\frac{v}{p} = 0 \tag{1.3}$$

$$\frac{u^2+v^2}{2} + \frac{\gamma}{\gamma-1}\frac{p}{\rho} = i_0(\eta), \quad p = \rho^\gamma f_0^\gamma(\eta)$$

Here u and v are the velocity components along the Ox- and Oy- axes, respectively; p is pressure; ρ is density; $\gamma > 1$ is the adiabatic exponent; f_0 and i_0 are arbitrary functions. The independent variables ξ (the function introduced by Martin [3]) and η (the stream function) are defined by

$$d\xi = \rho uv dy - (p + \rho v^2)\ dx, \ d\eta = \rho u dy - \rho v dx \tag{1.4}$$

The constants a and b in (1.1) and the variables ξ and η have the following dimensions:

$$[a] = L^{m+1}T^{-1}, \ [b] = ML^{-2-2m}, \ [\xi] = MT^{-2}, \ [\eta] = ML^{-1}T^{-1} \tag{1.5}$$

Therefore the only dimensionless parameter is the quantity $s = \xi a^{-2}b^{-1}$, and the functions in (1.3) can be written in the form

$$u = a^2b\eta^{-1}U(s), \ v = a^2b\eta^{-1}V(s), \ s = \xi a^{-2}b^{-1} \tag{1.6}$$

$$p = \eta^{-1/m}a^{(2m+1)/m}b^{(m+1)/m}P(s), \quad \rho = \eta^{(2m-1)/m}a^{(1-2m)/m}b^{(1-m)/m}R(s)$$

The dimensionless functions U, V, P, and R satisfy the system of equations obtained from (1.3),

$$\frac{1}{P} + m\left(\frac{V}{P}\right)_s' = 0, \quad V\left(\frac{V}{U}\right)_s' + P\left(\frac{1}{RU}\right)_s' = 0$$

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$$\frac{U^2+V^2}{2}+\frac{\gamma}{\gamma-1}\frac{P}{R}=H^2, \quad P=(RC_1)^{\gamma} \tag{1.7}$$

Here H and C_1 are arbitrary constants (the functions f_0 and i_0 are defined to within a constant factor).

The stream function η does not undergo a discontinuity upon passing through the shock. Therefore

$$d\eta = \rho_0 u_0 dy = aby^{m-1}dy$$

Hence

$$y^m = m\eta(ab)^{-1}$$

Since the image of the shock in the $\xi\eta$-plane is the line s = 0 [2], we write conditions (1.2) for the functions U, V, P, and R in the form

$$U|_{s=0}=\frac{\gamma-1}{(\gamma+1)m}, \quad V|_{s=0}=0$$

$$R|_{s=0}=\frac{\gamma+1}{\gamma-1}m^{(2m-1)/m}, \quad P|_{s=0}=\frac{2}{\gamma+1}m^{-1/m} \tag{1.8}$$

Solving (1.7) and using (1.8), we find

$$U=\frac{1}{m}-\frac{\gamma+1}{\gamma-1}m\tau, \quad V=\left(\frac{2}{\gamma-1}\tau-\left(\frac{\gamma+1}{\gamma-1}\right)^2 m^2\tau^2\right)^{1/2}$$

$$P=\tau R, \quad R=\frac{\gamma+1}{\gamma-1}m^{(2m-1)/m}\left(\frac{\tau}{\tau_1}\right)^{\gamma/(\gamma-1)}, \quad \tau_1=\frac{2(\gamma-1)}{(\gamma+1)^2 m^2} \tag{1.9}$$

$$s=m\frac{\gamma+1}{(\gamma-1)^2}\int_{\tau_1}^{\tau}\frac{1}{V(\tau)}\left(1-\frac{\gamma+1}{\gamma-1}m^2\tau\right)d\tau$$

The equations of the streamlines behind the shock can be written in parameteric form (taking the shock as the Oy-axis)

$$x=-\left(\frac{m}{ab}\right)^{1/m}h\eta^{1/m}\int_{\tau_1}^{\tau}\frac{\gamma-1-(\gamma+1)m^2z}{\sqrt{(\gamma-1)2z-(\gamma+1)^2m^2z^2}}z^{-\gamma/(\gamma-1)}dz, \quad h=\frac{\tau_1^{\gamma/(\gamma-1)}}{(\gamma-1)m}$$

$$y=-\left(\frac{m}{ab}\right)^{1/m}hm\eta^{1/m}\int_{\tau_1}^{\tau}\frac{\gamma-1-(\gamma+1)m^2z}{\sqrt{(\gamma-1)^2-2(\gamma^2-1)m^2z+(\gamma+1)^2m^4z^2}}z^{-\gamma/(\gamma-1)}dz+\left(\frac{m}{ab}\right)^{1/m}\eta^{1/m} \tag{1.10}$$

Thus, (1.6), (1.9), and (1.10) completely describe the gas flow behind a normal shock.

Directly behind the shock V (τ_1) = 0. Therefore, with motion along any streamline η = const downstream from the shock τ decreases from τ_1 (at the shock) to 0, and V → 0, x, y → ∞ as $\tau \to 0$. We denote by α the slope of the streamlines to the Ox-axis. Behind the shock $\alpha = 0$, as we move downstream α increases to some value α_{max} and then decreases to 0. It is not difficult to show using (1.9) that the value α_{max}, corresponding to the inflection point on the streamline, is independent of η and is reached for $\tau=\tau_*$, where

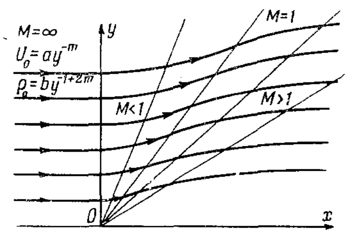

Fig. 1

$$\tau_*=\frac{\gamma-1}{\gamma(\gamma+1)m^2}, \quad \operatorname{tg}\alpha_{max}=\frac{1}{\sqrt{\gamma^2-1}}$$

We further find from (1.9)

$$M^2=\frac{u^2+v^2}{\gamma p}\rho=\frac{\gamma-1-2\gamma m^2\tau}{\gamma(\gamma-1)m^2\tau}, \quad M(\tau_*)=1$$

Hence, and also from (1.10), we see that the lines along which M = const are straight lines passing through the coordinate origin. Specifically, the sonic line (M = 1) of the flow behind the shock is a straight line and the locus of points of inflection of the streamlines behind the shock (Fig. 1). Taking any line η = const as the wall, we obtain the flow past a curvilinear contour.

2. Let us assume that a gas having parameters (1.1) passes through an oblique rectilinear shock.

Using the known relations for a strong oblique shock [1], we can show that the flow behind the shock is described as before by solution (1.9), in which τ_1 is to be replaced by $\tau_1 \sin^2\beta$, where β is the slope of the shock to the Ox-axis. This means that in the flow shown in the figure any straight line y = kx can be taken as the oblique shock.

On the basis of the solution obtained here and using the substitution principle [4], we can construct non-self-similar adiabatic flows containing shocks.

LITERATURE CITED

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